647510 **MSC4MatC401x** **Seat No :________**

M.Sc.(Maths.) Semester - 4 (*CBCS*) Examination
April/May-2023 (NEW COURSE)
Linear Algebra (Core)

Time: 2:30 Hours **Marks: 70**

Instructions:

1. All questions are compulsory.
2. Figures to the right indicate marks.

Q.1(A) Answer the following questions. (Any Two) **(10)**

1. If $\mathcal{A}$ is an algebra over F with unit element then $\mathcal{A}$ is isomorphic to a subalgebra of A(V) for some vector space V over F.
2. Let V is an n-dimensional vector space over F. If $\{v_1, v_2, \dots, v_n\}$ is a basis of V over F, if $T \in A(V)$ and $m(T)$ is the matrix of T in the basis $\{v_1, v_2, \dots, v_n\}$ then the mapping $\emptyset : A(V) \to M_n(F)$ defined as $\emptyset(T) = m(T)$ is an algebra homomorphism.
3. Let V be a finite dimensional vector space F and $T \in A(V)$. If $\lambda_1, \lambda_2, \dots, \lambda_k$ in F are distinct characteristic roots of T and $v_1, v_2, \dots, v_k$ are characteristic vectors of T corresponding to $\lambda_1, \lambda_2, \dots, \lambda_k$ respectively, then $v_1, v_2, \dots, v_k$ are linearly independent.

Q.1(B) Answer the following question. (Any One) **(04)**

1. Let V be a finite dimensional vector space over F. If $T \in A(V)$ is singular if and only if there exists a $v \in V, v \neq \theta$ such that $T(v) = \theta$.
2. Let T : $\mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $T(x, y, z) = (-2x + y, -y - z, x + 3z)$. Find the matrix of T with respect to the basis $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

Q.2(A) Answer the following questions. (Any Two) **(10)**

1. Let V be a vector space over F and let T $\in A(V)$ be nilpotent with the index n_1. Let $v \in V$ be such that $T^{n_1 - 1}(v) \neq \theta$.
   Let $V_1 = L(\{v, Tv, T^2v, \dots, T^{n_1-1}v\})$. Then dimension of V_1 is n_1, V_1 is invariant under T and the matrix of $T|_{V_1}$ is m_{n_1}.
2. Let V be a finite dimensional vector space over F and T ∈ A(V) be nilpotent. Then the invariants of T are unique.
3. Let V be a finite dimensional vector space over F and T ∈ A(V). If all the characteristic roots of T are in F then there is a basis of V with respect to the matrix of T is triangular.

Q.2(B) Answer the following question. (Any One) (04)

1. Let V be an n-dimensional vector space over F. If T ∈ A(V) has all its characteristic roots of T are in F, then T satisfies a polynomial of degree n over F.
2. Let V be a finite dimensional vector space over F and T ∈ A(V). Let W be a subspace of V which is invariant under T. Then T induces a map $\bar{T} = V/W \to V/W$ defined by $\bar{T}(v + W) = Tv + W,\ v \in V$ such that $\bar{T}$ well defined and $T \in (V/W)$.

Q.3(A) Answer the following questions. (Any Two) (10)

1. State and prove Jordan Decomposition theorem.
2. Let V be a finite dimensional vector space over F and $T \in A(V)$. Suppose that $V = V_1 + V_2,$ where V_1 and V_2 are subspaces of V invariant under T. Let $T_1 = T|_{V_1}$ and $T_2 = T|_{V_2.}$ If the minimal polynomial of T_1 over F is $p_1(x)$ and the minimal polynomial of T_2 over F is $p_2(x)$ then the minimal polynomial of T over F is the least common multiple of $p_1(x)$ and $p_2(x)$.
3. Let $T \in A(V)$, has minimal polynomial over F is $p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_{r-1} x^{r-1} + x^r$. Further suppose that V is a cyclic $F[x]$ module then there is basis of V over F such that , in this basis the matrix of T is $C\big(p(x)\big)$.

Q.3(B) Answer the following question. (Any One) (04)

1. Define Jordan block belonging to λ and write the Jordan blocks of the minimal polynomial $p(x) = \ (x-2)^3(x+4)^2$.
2. Define companion matrix of $f(x) \in F[x]$. Write the companion matrix of the polynomials $f_1(x) = 1 + 2x + x^3$ and $f_2(x) = 2 + 5x + 3x^2 + 7x^3 + x^4$.

Q.4(A) Answer the following questions. (Any Two) (10)

1. If two rows of $A \in M_n(F)$ are equal, then show that $\det(A) = 0$.
2. State and prove Cramer's rule for system of n linear equations.
3. Prove that Hermitian linear transformation T is nonnegative if and only if all of its characteristic roots are nonnegative.

Q.4(B) Answer the following question. (Any One) (04)

1. Let F be a field. Then prove the followings for $A, B \in M_n(F), \lambda \in F$.
   (I) $\mathrm{tr}(\lambda A) = \lambda\ \mathrm{tr}(A)$ **(II)** $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$,
   (III) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
2. Let F be a field of characteristic 0 and V be a vector space over F. If $S, T \in A(V)$ such that $ST - TS$ commutes with S, then $ST - TS$ is nilpotent.

Q.5(A) Answer the following questions. (Any Two) **(10)**

1. Let V be a finite dimensional vector space over a field of characteristic 0, and let f be a symmetric bilinear form on V. Then there is an ordered basis for V in Which f is represented by a diagonal matrix.
2. Let $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a map. Then f is bilinear if and only if there exist $\alpha_{ij} \in \mathbb{R}, 1 \le i, j \le n$ with $\alpha_{ij} = \alpha_{ji}$ such that
$$f(x, y) = \sum_{i,j=1}^{n} \alpha_{ij} x_i y_j.$$
3. Let f be a non-degenerate bilinear form on a finite dimensional vector space V. The set of all linear operators on V which preserve f is a group under the operation of composition.

Q.5(B) Answer the following question. (Any One) **(04)**

1. Find the symmetric matrices associated with the following quadratic forms:
   (I) $9x_1^2 - x_2^2 + 4x_3^2 + 6x_1x_2 - 8x_1x_3 + 2x_2x_3$
   (II) $xy + yz + xz$
   (III) $x_1^2 + x_2^2 - x_3^2 - x_4^2 + 2x_1x_2 - 10x_1x_4 + 4x_3x_4$
   (**IV**) $-y^2 - 2z^2 + 4xy + 8xz - 14yz.$
2. Determine the rank and signature of the quadratic form
   $3x^2 + 5y^2 + 3z^2 - 2xy + 2xz - 2yz.$

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